Ordered Blueprints

Def: An ordered blueprint is a triple $(B^\bullet, B^+, \leq)$, where:

* B^+ is a semiring;

* $B^\bullet \subseteq B^+$ is a multiplicative subset containing 0 and 1 that generates B^+ as semiring (i.e., every element of B^+ is a finite sum of elements in $B^\bullet$);

* $\leq$ is a partial order on B^+ satisfying

$$\left.\begin{array}{l} x \leq y \\ c \in B^+ \end{array}\right\} \Rightarrow \left\{\begin{array}{l} cx \leq cy \\ c+x \leq c+y \end{array}\right.$$

Rmk: B^+ is a pointed and commutative monoid

- A morphism $f: B \longrightarrow C$ of ordered blueprints is a morphism $f^+: B^+ \longrightarrow C^+$ of semirings that respects the order and satisfies $f^+(B^\bullet) \subseteq C^\bullet$

So one has the category $\mathcal{OBl}_{pr}$ of ordered blueprints.

Some examples

1) The trivial ordered blueprint $\underline{0} = (\{0\}, \{0\}, =)$ is the final object in OBl_{pr}.

2) $\underline{1} := (\{0,1\}, \mathbb{N}, =)$ is initial in OBl_{pr}.

3) $\mathbb{F}_{1^n} := (\mu_n \sqcup \{0\}, S, =)$, where

$\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$, $S := \mathbb{Z}[\mu_n] \subseteq \mathbb{C}$ the subring of $\mathbb{C}$ generated by the group μ_n

Note that there exists a morphism $\mathbb{F}_{1^n} \longrightarrow \mathbb{F}_{1^m}$ in $\mathcal{OBl}_{pr}$ that is injective iff $n \mid m$ and $\mathrm{Aut}_{\mathcal{OBl}_{pr}}(\mathbb{F}_{1^n}) \simeq \mathrm{Aut}_{\mathrm{Mon}_*}(\mu_n \sqcup \{0\}) \simeq \Big(\underbrace{\mathbb{Z}/_{n\mathbb{Z}}}_{\text{ring}}\Big)^\times$

Rmk: $\mathbb{Z} \hookrightarrow \mathbb{Z}[\mu_3]$ is a morphism of semirings but not a morphism $\mathbb{F}_{1^2} \longrightarrow \mathbb{F}_{1^3}$ in $\mathcal{OBl}_{pr}$ $(-1 \notin \mu_3)$

4) There is a fully faithful functor from semirings to ordered blueprints

$$\mathcal{F} : \mathrm{Srings} \longrightarrow \mathrm{OBlpr}$$

$$R \longmapsto (R^\bullet, R^+, =), \quad R^\bullet = R \text{ with mult.}$$
$$R^+ = \text{semiring } R$$

$$f : R \to S \longmapsto f : \mathcal{F}R \to \mathcal{F}S$$
(same function)

Rmk: $\underline{0} = \mathcal{F}(\{0\})$

There exists a faithful functor

$$\tilde{\mathfrak{F}} : \mathrm{OBl}_{pr} \longrightarrow \mathrm{Sring}$$

$$B \longmapsto B^+$$

$$g : B \to C \longmapsto g : B^+ \longrightarrow C^+$$

<u>that is not full</u>: $f : \mathbb{Z} \hookrightarrow \mathbb{Z}[\mu_3]$ is in $\mathrm{Hom}_{\mathrm{Sring}}(\tilde{\mathfrak{F}}\mathbb{F}_{1^2}, \tilde{\mathfrak{F}}\mathbb{F}_{1^3})$ but not in $\mathrm{Hom}_{\mathrm{OBl}_{pr}}(\mathbb{F}_{1^2}, \mathbb{F}_{1^3})$

Note that $\tilde{\mathfrak{F}}\mathfrak{F} = \mathrm{Id}_{\mathrm{sRing}}$,

but $\mathfrak{F} \not\dashv \tilde{\mathfrak{F}}$, because

$$\emptyset = \mathrm{Hom}_{\mathrm{OBl}_{\mathrm{pr}}}(\mathfrak{F}\mathbb{N}, 1)$$

$$\text{and } \mathrm{Hom}_{\mathrm{sRing}}(\mathbb{N}, \overbrace{\mathfrak{F}1}^{\mathbb{N}}) = \{\mathrm{id}_{\mathbb{N}}\}$$

nor $\tilde{\mathfrak{F}} \dashv \mathfrak{F}$: Let $B := (\mathbb{N}, \mathbb{N}, \leq)$ usual $n \leq n+1$

Then $\{\mathrm{id}_{\mathbb{N}}\} = \mathrm{Hom}_{\mathrm{sring}}(\tilde{\mathfrak{F}}B, \mathbb{N})$

$$\text{and } \mathrm{Hom}_{\mathrm{OBl}_{\mathrm{pr}}}(B, \mathfrak{F}\mathbb{N}) = \emptyset$$

5) $\exists$ fully faithful functor from pointed commutative monoids too:

$$\mathcal{G}: \mathrm{Mon}_* \longrightarrow \mathrm{OBl}_{pr}$$

formal finite sums of elements in $M \setminus \{0\}$

$$M \longmapsto (M, \mathbb{N}[M\setminus\{0\}], =)$$

$(\mathbb{N}[\emptyset] := \{0\})$

$$\varphi: M \to W \longmapsto \varphi^+ : \mathbb{N}[M\setminus\{0\}] \longrightarrow \mathbb{N}[W\setminus\{0\}]$$

$$\sum m_i \longmapsto \sum \varphi(m_i)$$

Rmk: $\underline{1} = \mathcal{G}(\{0,1\})$

As in the case of semirings:

$$\exists\ \tilde{\mathcal{G}} : \mathrm{OBl}_{pr} \longrightarrow \mathrm{Mon}_* \quad \text{faithful}$$

$$B \longmapsto B^{\bullet} \qquad (B^{\bullet} \text{ generates } B^{+})$$

$$\begin{aligned} g : B \longrightarrow C \longmapsto g^{\bullet} : B^{\bullet} &\longrightarrow C^{\bullet} \\ b &\longmapsto g(b) \end{aligned}$$

that is not full: $\mathrm{Hom}_{\mathrm{OBl}_{pr}}\left(\mathcal{F}(\mathbb{Z}/2\mathbb{Z}), \mathbb{1}\right) = \emptyset$

but $\tilde{\mathcal{G}}\mathcal{F}(\mathbb{Z}/2\mathbb{Z}) = \{\bar{0}, \bar{1}\}$, $\tilde{\mathcal{G}}\mathbb{1} = \{0, 1\}$

and $\exists\ \{\bar{0}, \bar{1}\} \longrightarrow \{0, 1\}$ in Mon_*

$$\bar{x} \longmapsto x$$

Note that $\tilde{\mathcal{G}}\mathcal{G} = \mathrm{Id}_{\mathrm{Mon}_*}$ and

$\mathcal{G} \dashv \tilde{\mathcal{G}}$

$$
\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{OBl}_{pr}}(\mathcal{G}M, \mathcal{G}B^\bullet) \arrow[dl, "\pi\circ\_"'] \\
\mathrm{Hom}_{\mathrm{OBl}_{pr}}(\mathcal{G}M, B) \arrow[r, shift right, "\tilde{\mathcal{G}}"'] & \mathrm{Hom}_{\mathrm{Mon}_*}(M, \overbrace{\tilde{\mathcal{G}}B}^{=B^\bullet}) \arrow[l, dashed, shift right, "\sim"'] \arrow[u, "\mathcal{G}"']
\end{tikzcd}
$$

because $\exists!\, \mathbb{N}[B^\bullet \setminus \{0\}] \xrightarrow{\pi} B^+$ in Sring that fixes $B^\bullet$ and $B^\bullet$ generates B^+

But $\tilde{\mathcal{G}} \not\dashv \mathcal{G}$:

Let $C := (\{0,1\}, \mathbb{N}, \leq)$ usual $n \leq n+1$

$$\{\mathrm{id}_{\{0,1\}}\} = \mathrm{Hom}_{\mathrm{Mon}_*}(\tilde{\mathcal{G}}C, \{0,1\})$$

and $\mathrm{Hom}_{\mathrm{OBlpr}}(C, \mathcal{G}(\{0,1\})) = \emptyset$

6) Tropical semifield:

$$\mathbb{R}^{\max}_{\geq 0} := (\mathbb{R}_{\geq 0}, (\mathbb{R}_{\geq 0}, \max, \cdot), =)$$

usual mult.

"the sum is max"

$$\simeq \overline{\mathbb{R}} := ((\mathbb{R} \sqcup \{-\infty\}, +), (\mathbb{R} \sqcup \{-\infty\}, \max, +), =)$$

$$\mathbb{R} \sqcup \{-\infty\}, \max, + \;=\; \mathfrak{J}(\overline{\mathbb{R}})$$

$$\overline{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^{\max}_{\geq 0}$$

$$r \longmapsto e^r$$

Quotients

Def: Let B^+ be a semiring

A pre-order $\tau \, (\subseteq B^+ \times B^+)$ on B^+ is

- additive if $(x,y) \in \tau \Rightarrow (x+z, y+z) \in \tau$
- multiplicative if $(x,y) \in \tau \Rightarrow (xz, yz) \in \tau$

Notation: $x \leq_\tau y := (x,y) \in \tau$.

$$x \equiv_\tau y := x \leq_\tau y \text{ and } y \leq_\tau x$$

- Let $B = (B^\bullet, B^+, \leqslant_B) \in \mathrm{OBl}_{pr}$

For $S \subseteq B^+ \times B^+$, define:

$$\langle S \rangle := \bigcap \left\{ \tau \;\middle|\; \begin{array}{l} \tau \text{ is an additive and mult.} \\ \text{pre-order containing } S \text{ and } \leqslant_B \end{array} \right\}$$

the add. and mult. pre-order generated by S.

Prop: $\exists\, B/\!/\langle S\rangle \in \mathrm{OBl}_{pr}$ and a morphism $\pi : B \longrightarrow B/\!/\langle S\rangle$ s.t.

$\forall f : B \longrightarrow C$ with $f(x) \leq_C f(y)\ \forall (x,y) \in S$

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & C \\
B/\!/\langle S\rangle \arrow[ur, dashed, "\exists!\,\overline{f}"'] &
\end{tikzcd}$$

Idea of the proof

$\equiv_{\langle S \rangle}$ is an equivalence relation

$\overline{B}^+ := B^+/\equiv_{\langle S \rangle}$ is a semiring

$\pi : B^+ \twoheadrightarrow \overline{B}^+$ ⟵ the canonical map

$\overline{B}^\bullet := \pi(B^\bullet)$

$\pi(x) \leq_{\overline{B}} \pi(y) :\Leftrightarrow x \leq_{\langle S \rangle} y$

$$\boxed{B/\!/\langle S \rangle := (\overline{B}^\bullet, \overline{B}^+, \leq_{\overline{B}})}$$

Notation: • If $S = \{(x_i, y_i) \mid i \in I\}$, $\langle x_i \leq y_i \mid i \in I \rangle := \langle S \rangle$

• For $M \in \mathrm{Mon}_*$, $M /\!\!/ \langle S \rangle := \mathcal{G}(M) /\!\!/ \langle S \rangle$

Examples:

7) $\mathbb{F}_1^{\pm} := \{0, 1, \varepsilon\} /\!\!/ \langle 0 \leq 1 + \varepsilon \rangle$, where $\varepsilon^2 = 1$

8) for $a,b \in \mathbb{R}_{\geq 0}$, let

$$a \boxplus b := \begin{cases} \{\max\{a,b\}\}, & \text{if } a \neq b \\ [0,a], & \text{if } a = b \end{cases} \quad \text{(hyperaddition)}$$

The tropical hyperfield is the ordered blueprint:

$$\mathbb{T} := \mathbb{R}_{\geq 0} /\!/ \langle 1_{\mathbb{N}} \cdot c \leq 1_{\mathbb{N}} \cdot a + 1_{\mathbb{N}} \cdot b \mid c \in a \boxplus b \rangle$$

Let K be a field and $v : k \longrightarrow \mathbb{R}_{\geq 0}$ a nonarchimedean non-trivial absolute value.

Let $k^{mon} := \mathcal{G}(k) /\!\!/\!\!/ \langle 1_{\mathbb{N}} \cdot c \leq 1_{\mathbb{N}} \cdot a + 1_{\mathbb{N}} \cdot b \mid c = a + b \in k \rangle$

Rmk: $(k^{mon})^{\cdot} = k$, $(k^{mon})^{+} = \mathbb{N}[k \setminus \{0\}]$

$\mathbb{T}^{\cdot} = \mathbb{R}_{\geq 0}$, $\mathbb{T}^{+} = \mathbb{N}[\mathbb{R}_{>0}]$

Then $\exists ! \, \overline{v} : k^{mon} \longrightarrow \mathbb{T}$ morphism in $\mathcal{OB}l_{pr}$

$$\sum 1_{\mathbb{N}} \cdot \alpha_i \longmapsto \sum 1_{\mathbb{N}} \cdot v(\alpha_i)$$

Free-Algebras

Def: Let $B \in \mathrm{OBl}_{pr}$ and A a monoid (not necessarily pointed) We define the free ordered blue B-algebra in A, $B[A]$ where

$$B[A]^+ := \Big\{ \sum_{\text{finite}} x_a \cdot a \;\Big|\; x_a \in B^+,\ a \in A \Big\}$$

$$B[A]^\bullet := \{ b \cdot a \mid b \in B^\bullet,\ a \in A \}$$

$$\leq_{B[A]} := \langle x\cdot 1_A \leq y\cdot 1_A \mid x,y\in B^+ \text{ and } x\leq_B y\rangle$$

- $B[A]$ satisfies the following:

$$\forall \begin{cases} f: B \longrightarrow C & \text{of ordered blueprints} \\ \varphi: A \longrightarrow C^\bullet & \text{of monoids} \end{cases}$$

$$\begin{tikzcd}
x \arrow[d, mapsto] & B \arrow[r, "f"] \arrow[d] & C \\
x\cdot 1_A & B[A] \arrow[ur, dashed]
\end{tikzcd}
\quad \exists!\, \tilde{\varphi} \text{ s.t. } \tilde{\varphi}(1_B\cdot a)=\varphi(a) \ \forall a\in A$$

Examples:

9) B any ordered blueprint

$A := \{ T_1^{\alpha_1} \cdots T_n^{\alpha_n} \mid \alpha_i \in \mathbb{N} \}$ (the free monoid in n variables)

$B[T_1, \dots, T_n] := B[A]$

The polynomial blue B-algebra in n variables

10) Let M be a monoid.
Then $M \sqcup \{0\}$ is a pointed monoid and

$$\mathbb{1}[M] = \mathcal{G}(M \sqcup \{0\})$$

Ideals

Def: Let $B = (B^\bullet, B^+, \leq)$ be a blueprint,

* An m-ideal of B is an ideal I of the monoid $B^\bullet$ ($I \neq \emptyset$ and $xy \in I$ $\forall x \in B^\bullet, y \in I$)
* An ideal of B is an m-ideal I s.t.

$$\left.\begin{array}{l} b \in B^\bullet \\ a_1, \dots, a_n \in I \\ b = \sum a_i \in B^+ \end{array}\right\} \Rightarrow b \in I$$

* A k-ideal of B is an m-ideal I s.t.

$$\left.\begin{array}{l} c \in B^{\bullet} \\ a_1, \dots, a_n, b_1, \dots, b_m \in I \\ \sum a_i + c = \sum b_j \in B^{+} \end{array}\right\} \Rightarrow c \in I$$

An (m-/k-) ideal I is:

- proper if $I \neq B^\bullet$;
- prime if $B^\bullet \setminus I$ is a multiplicative non-empty set;
- maximal if I is proper and $\nexists J$ proper (m-/k-) ideal s.t. $I \subsetneq J \subsetneq B^\bullet$.

Some properties

1. Let $f: B \longrightarrow C$ be a morphism in OBlpr and I an (m-/k-) (proper/prime) ideal of C.

Then $f^{-1}(I) \cap B^{\bullet}$ is an (m-/k-) (proper/prime) ideal of B.

2. $\{k\text{-ideals}\} \subseteq \{\text{ideals}\} \subseteq \{m\text{-ideals}\}$

3. $\forall B \in \mathrm{OBl}_{pr}$ has a unique maximal m-ideal: $B^\bullet \setminus \underbrace{(B^\bullet)^\times}_{\text{the group of invertible elements}}$

4. Let $S \subseteq B^\bullet$.

Then the intersection of all (m-/k-) ideals of $B^\bullet$ containing S is an (m-/k-) ideal of $B^\bullet$

Notation: $\langle S \rangle$, $\langle S \rangle_m$, $\langle S \rangle_k$

Rmk: If $S \subseteq B^\bullet \rightsquigarrow \langle S \rangle$ is an ideal
If $R \subseteq B^+ \times B^+ \rightsquigarrow \langle R \rangle$ is a pre-order

Def: For $f: B \to C$ in $OB|_{pr}$, let $\ker(f) := f^{-1}(\{0\}) \cap B^\bullet$ the kernel of f.

5. * $\ker(f)$ is a k-ideal

* $\langle S \rangle_k = \ker\left(B \longrightarrow B /\!/ \langle 0 \equiv x \mid x \in S \rangle\right)$

$= \{ y \in B^\bullet \mid \exists a_i\text{s}, b_j\text{s} \in S \text{ s.t. } \sum a_i + y = \sum b_j \}$

Examples

11) If R is a semiring and $B = \mathcal{I}(R)$, then: (k-)ideal of B = (k-)ideal of R

12) B^+ ring $\Rightarrow$ ideal = k-ideal

13) Let $B = \mathcal{P}(\mathbb{N})$

Then $\mathbb{N} \setminus \{1\}$ is an ideal, but not a k-ideal of B

$(2, 3 \in \mathbb{N} \setminus \{1\},\ 2+1=3$ and $1 \notin \mathbb{N} \setminus \{1\})$

Localization

Def: Let $B \in \mathrm{OBl}_{pr}$ and $1 \in S \subseteq B^\bullet$ mult. set

We will construct the localization of B w.r.t. S as an ordered blueprint: $S^{-1}B$.

- The monoid: $(S^{-1}B)^\bullet := S^{-1}(B^\bullet)$ ← localization of monoids

$S^{-1}(B^\bullet) = B \times S /\sim$, $(a,s) \sim (b,t) :\Leftrightarrow \exists w \in S$ s.t. $wat = wbs$ in $B^\bullet$

- The semiring: $(S^{-1}B)^{+} := S^{-1}(B^{+})$ as before

The + operation works as in the case of rings: the equivalence class of (b,t)

$$\frac{a}{s} + \frac{b}{t} := \frac{at+bs}{st}$$ and $$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

- The order:

$$\frac{c}{t} \leqslant \frac{h}{p} := \exists a \in S \text{ s.t. } cap \leqslant_B hat$$

- This localization satisfies the expected universal property:

<u>Prop</u>: $\forall_f$ in OBlpr

$$
\begin{tikzcd}
\alpha \arrow[d, mapsto] & B \arrow[r, "f \text{ s.t. } f(S) \subseteq (C^\cdot)^\times"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \\
\frac{\alpha}{1} & S^{-1}B \arrow[ur, dashed] &
\end{tikzcd}
$$

$\exists !\, \tilde{f} : \frac{a}{s} \longmapsto f(a) f(s)^{-1}$

Examples

14) Let R be a semiring and $S \subseteq R$ mult. set. Then $S^{-1}(\mathcal{F}R) = \mathcal{F}(S^{-1}R)$

15) Let B be an ordered blueprint.
Let $M := \{T_1^{\alpha_1} \cdots T_n^{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \mathbb{N}\}$
$S := M$
Then $B[T_1^{\pm 1}, \dots, T_n^{\pm 1}] := S^{-1}(B[M]) = B[S^{-1}M]$

16) If P is a prime m-ideal of B, then $B_P := (B^{\cdot} \setminus P)^{-1} B$

Note that the maximal m-ideal of B_P is $(B^{\cdot} \setminus P)^{-1} P = \left\{ \frac{a}{s} \mid a \in P, s \notin P \right\}$